

The Lee Fields Medal III

TIME ALLOWED: UP TO THREE HOURS

TABLES AND CALCULATORS MAY BE USED.

ANSWER ALL TEN QUESTIONS

1. Does there exist a rectangle of area 401 with whole-number sides? Justify your answer.

2. A real number that can be written in the form

$$\frac{m}{n},$$

with m and n whole numbers is a *rational* number, while one which cannot is called an *irrational* number. Examples of irrational numbers include $\sqrt{2}$, e , and π .

(a) Given that $\sqrt{2}$ is irrational, show that $1 - \sqrt{2}$ is irrational.

(b) Hence, or otherwise, write down two irrational numbers whose sum is rational.

3. Suppose that $m > n$ are whole numbers. Show that $m^2 + n^2 > 2mn$. Furthermore, show that a triangle with side lengths $m^2 + n^2$, $m^2 - n^2$, $2mn$ is a right-angled-triangle.

4. Consider the three lines:

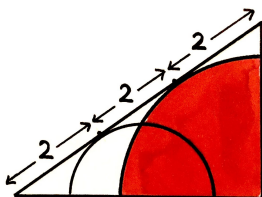
$$l_1 : y = 3x + 1$$

$$l_2 : y = x + 2$$

$$l_3 : y = -2x + 6$$

Does the intersection $l_1 \cap l_2 \cap l_3$ contain any points? Justify your answer.

5. Find the area of the quarter-circle.



6. With the aid of a diagram, or otherwise, prove that for $0^\circ < \theta < 90^\circ$

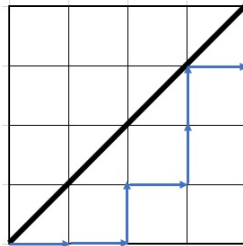
$$\sin \theta + \cos \theta > 1.$$

7. Where d is the outcome of a rolled dice, consider the quadratic function:

$$q(x) = x^2 + 3x + d.$$

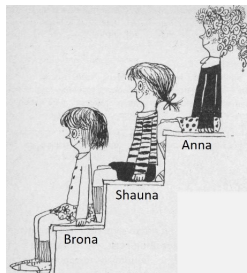
What is the probability that q has real roots?

8. A student is in the center of a square pool, the teacher at the corner. Teacher runs three times as fast as student swims, but the student runs faster than the teacher. Can the student escape past the teacher?
9. The n -th *Catalan number* is equal to the number of *monotonic lattice paths* along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. Here is an example of a monotonic lattice path which does not pass above the diagonal in a 4×4 grid:



Find the fourth Catalan number by finding the number of such paths in a 4×4 grid.

10. Three friends, Anna, Brona and Shauna, are seated in a lecture hall. Shauna can only see Brona, Anna can see both girls seated in front of her.



In a box there are 2 white hats and 3 black hats. Three hats are taken out of the box and are put on the three girls. The girls did not see what hats they were given and do not know the color of the hats left in the box. When Anna was asked about the color of her hat she said she could not answer. When Shauna heard Anna's answer, she also said that she could not figure out the color of her hat. Can Brona, based on the other girls' answers, figure out the color of the hat she is wearing?